THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE INTERESTS OF TEACHERS OF MATHEMATICS

Volume I June, 1909 Number 4

CONTENTS

Published Quarterly by the

ASSOCIATION OF TEACHERS OF MATHEMATICS FOR THE MIDDLE STATES AND MARYLAND

LANCASTER, PA., SYRACUSE, N. Y.

NEW YORK, N. Y., PHILADELPHIA, PA.

THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

THE MATHEMATICS TEACHER is published quarterly—September, December, March and June—under the auspices of the Association of Teachers of Mathematics for the Middle States and Maryland.

The annual subscription price is $1.00; single copies, 35 cents.

Remittances should be sent by draft on New York, Express Order or Money Order, payable to The Mathematics Teacher.

THE MATHEMATICS TEACHER,
41 NORTH QUEEN STREET, LANCASTER, PA.
OR SYRACUSE, N. Y.

THE MATHEMATICS TEACHER

EDITED BY

W. H. METZLER

ASSOCIATED WITH

EUGENE R. SMITH JONATHAN T. RORER

VOLUME I JUNE, 1909 NUMBER 4

THE TEACHING OF MATHEMATICS IN THE ELEMENTARY AND THE SECONDARY SCHOOL.*

By ARTHUR SULLIVAN GALE.

(*Continued from p. 119.*)

The chapter on geometry (pp. 257–291) is excellent. It presents ideas on geometric analysis, concrete and formal geometry, methods for treating problems, modern geometry, and non-Euclidean geometry. Especially important is the discussion of problems which lead to algebraic equations and the construction of simple algebraic expressions. A timely plea is made for experimental work and the use of models and apparatus. As an example of their value, a Rochester teacher exhibited a sextant before a class one morning. A pupil borrowed it for the noon hour and became so enthusiastic in its use that he "cut" his afternoon classes to do some rough surveying. Contrast the interest which the instrument developed with the lack of enthusiasm which causes so many absences from the mathematical class-room! The chapter closes with an analysis of trigonometry and suggestions as to where its various parts should be taught.

The chapter on algebra (pp. 292–326) practically begins with a discussion of the tendencies of the day based on the report of the committee appointed by the American Mathematical Society in 1902, and closes with a topical outline of an order of treatment in accordance with the ideas developed in the chapter.

* By J. W. A. Young. Longmans, Green and Co., 1907.

Much attention is given to the arithmetical side of algebra and to the interpretation of equations, a topic deserving more attention than it usually receives. The equation is discussed under the suggestive heading of "The back-bone of algebra" and many important points are well treated, especially the introduction of logarithms. A few points, of which we mention two, are open to criticism. To the reader who does not know how to solve a system of linearly dependent, non-homogeneous, linear equations, it is not enough to say of a system of inconsistent equations, used illustratively, that it "admits of no solution, as may be verified by attempting to solve it" (p. 305). Nor is this satisfactory: "Five is simply a number; it may be used as addend, subtrahend, multiplier, divisor or otherwise; $+5$ is a number to be added; -5 is a number to be subtracted" (p. 317).

The volume closes with a chapter on limits (pp. 327–346) including a discussion of irrational numbers and infinity. By means of an outline of a rigorous treatment, the difficulties of such a treatment are made very clear. The opinion that the theory of limits should be eliminated from secondary instruction seems to be rapidly becoming more prevalent.

The book is of great bibliographic value. Nearly every chapter is headed by an extensive list of texts and papers, while the pages themselves are exceptionally rich in foot-notes containing exceedingly interesting quotations, with references from a very wide range of authors. It is evident from the preface and also here and there throughout the book that other references might have been cited in various connections. The most noteworthy omission is that of the admirable volume edited by F. Enriques, Questioni riguardanti la geometria elementare, of which Teubner has published a much enriched translation into German. The value of Professor Young's book for reference is enhanced by a table of contents giving a complete analysis of each chapter by topics and sub-topics, and by a good index.

The book deserves a very wide circulation among teachers of mathematics, and will undoubtedly be a source of information and inspiration to the large number of teachers who are looking for the means to better their instruction.

THE UNIVERSITY OF ROCHESTER,
ROCHESTER, N. Y.

TEACHING CLASSES IN GEOMETRY TO SOLVE ORIGINAL EXERCISES.*

By Fletcher Durell.

We all realize, I presume, that we have a difficult problem before us. I feel that I shall be more likely to say something that you will care to listen to if I confine myself largely to personal experience. I will however give a little theory at the outset.

It has often been remarked that a pupil in studying geometry is learning to use a certain set of tools, viz.: the line, angle, triangle, circle, etc. In like manner it is stated that in learning to solve original exercises he has a kit or chest of tools before him and is learning to select for himself that geometric instrument or set of instruments which will accomplish a desired result. Thus when he attempts to prove that the bisectors of the opposite angles of a parallelogram are parallel, the pupil is taught to observe that he has on the figure a quadrilateral, two triangles, and various sets of parallel lines, and to recall the various theorems relating to each of these objects or sets of objects, and to test or to try each of these theorems in succession till he finds one or more which will enable him to prove the desired result. In this way it is possible to get several different proofs for the original exercise just mentioned.

But in practice this kit-of-tools process as ordinarily stated is found very fragmentary and defective. It surely is a loose and unsystematic method of procedure to take a mere catalogue of geometric objects and the theorems relating to them and to try these in additive succession till we strike one that will answer our purpose. Also in this process, the tendency is to limit our tests to those specific geometric concepts which are closely related to the matter in hand, such as parallel lines, the triangle, and parallelogram, and neglect more general, and hence on the whole more important and more powerful toolages such as the axioms, and proofs by superposition, or by negative methods. Also the use of auxiliary quantities, the most important and powerful tool in proving difficult originals and the one needing most careful control and the most systematic use, is left wholly to haphazard and accidental treatment.

It has occurred to me that it is important in this connection

* Read at a meeting of the Philadelphia Section.

to raise the question of the nature of a geometric tool. What is the inner essence or common property of all geometric instruments, and indeed of implements used in work of any kind? The answer seems to be that a geometric tool is something external to the data of a problem, just as a saw or axe is external to the mechanic using it. Proof is a step-by-step process. But it cannot be this unless we have something to work with. Without this, instead ot a proof we have immediate intuition. An instrument is a supplementary something used in an auxiliary way. Toolage means auxiliary quantity or entity. This latter idea is more general than that of a set of particular tools and includes any number of tools as particular cases of itself. Being general and continuous it should open the way to a more comprehensive classification and a more systematic treatment of originals in geometry.

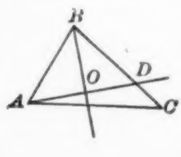
Let me present a specimen way in which I think the above idea can be made a source of progress in this department of mathematical pedagogy.

Geometric toolages, *i. e.:* Auxiliary quantities may be first classified as natural or artificial, that is, as that which is existent or given us in a problem or that which we create and introduce.

An example of a natural or existent auxiliary quantity is the third angle which is used in proving the theorem that two opposite or vertical angles formed by intersecting straight lines are equal. An example of artificial auxiliary quantity is the line drawn to bisect the vertical angle of an isosceles triangle in the proof of the theorem that angles at the base of an isosceles triangle are equal.

Again each of these forms of toolage may be subdivided into internal or external instrumentalism, that is, into auxiliary quantity internal or external in position with reference to the data of a problem. Combining these ideas we have the tools or toolage entities of geometry classified into four primary groups which we shall denote by A, B, C, D.

A. *Natural Internal Auxiliary Quantity.*—Thus if on the figure to the left we have given that ABC is any triangle, that BO bisects the angle ABC and AD is perpendicular to BO meeting BO at O and BC at D, and it is required to prove $\triangle ABO = \triangle DBO$; and if we prove the tri-



angles equal by means of the sides and angles which are internal to or a part of the triangles, we are using toolage of the species A.

B. *Natural External Auxiliary Quantity.*—An example of B occurs in proving the diagonals of a rectangle equal by means of a pair of triangles of which the diagonals named form a part or detail, the triangles being existent on the given figure but being in the main external to the given diagonals.

The third angle used in proving two opposite or vertical angles equal mentioned above in another illustration of species B.

C. *Artificial External Auxiliary Quantity.*—The auxiliary line used in proving the base angles of an isosceles triangle equal, also mentioned above, is an ample of C. We have another example in the auxiliary line or lines drawn within an isosceles trapezoid in the process of proving the base angles of an isosceles trapezoid equal. Of course innumerable examples of this as well as of the other species named occur throughout geometry.

D. *Artificial External Auxiliary Quantity.*—A familiar example of the D species is furnished by the auxiliary line or lines drawn outside of a triangle in proving the sum of the angles of a triangle equal to two right angles.

It is also important to make subdivisions of D. Artificial external quantity may be subdivided into forms which are (1) Finite or (2) infinite.

An instance of the finite variety of D is the circumference drawn to circumscribe a regular polygon in proving properties of the polygon. An instance of the infinite variety of D is the use of three-dimensional space when in a proof by superposition a geometric object is picked up out of a plane and turned over and put back in the plane. Also this infinite variety of D may take two main forms, (*a*) the geometric form as in the use of three-dimensional space just referred to, or in proof by use of an unlimited locus, (*b*) the logical infinite such as is used in negative demonstrations as when we prove one object greater than another by proving that it is not equal to it or less than it. Here we obtain our proof by the use of an infinite logical negative matrix.

The differences in the above general A, B, C, D methods of proof find an illustration in the various possible proofs of the theorem that the diagonals of a regular pentagon are equal.

If we prove this property by means of a pair of triangles formed by the diagonals and the sides of a pentagon, we use B; if by dropping perpendiculars from the vertices of the pentagon upon the diagonals, we use C; if we obtain a proof by circumscribing a circle about the given regular pentagon we use D, 1; if by superposition of triangles, we use D, 2.

It should also be observed at this point that the axioms find a natural place in the above schematism. For the axioms are seen to be special and important cases of A, B, C, D. Thus the axiom that a whole is equal to the sum of its parts, falls under A; the axiom that things equal to the same thing are equal to each other, falls under B, or D, etc. Hence, to our A, B, C, D we may add E, the axioms.

It is also recognized, of course, that the above special forms or ways of applying auxiliary objects or entities may be combined and compounded variously.

It may be well to tabulate the results of our analysis. We have:

I. *General Geometric Toolages.*
- A. Natural Internal
- B. Natural External
- C. Artificial Internal
- D. Artificial External
  - 1. Finite
  - 2. Infinite
    - (*a*) Geometric
      Ex. Proof by superposition or by the locus
    - (*b*) Logical
      Ex. Proofs by negative methods
- E. The axioms (= important special cases of A, B, C, D, and are broader than geometry).

II. *Specific Toolages.*
- X. Geometric objects considered
- Y. Known theorems concerning these geometric objects, considered singly or in combination.

I will not at this place try to make a complete statement of the virtues or defects of the above method of classifying the instruments used in geometric proofs. I will, however, point out such of its qualities as now open the way to a further development of the plan.

1. This A, B, C, D method proceeds from the simple to the complex and in particular opens the way to check the pupil and especially the weak pupil, in his wild tendency to use all sorts of auxiliary lines. In order to be able to control this fatal tendency

it is well to note its probable cause. This tendency on the part of the pupil probably arises from the fact that auxiliary quantities of the C and D types are visible and palpable and hence seem to hold out larger promises of help to the pupil while auxiliary objects of the A and B form are often implicit and a matter of relations which do not strike the eye vividly. Hence we realize the prime importance of holding pupils to the A and B methods of proof till these are thoroughly mastered before C and D are taken up.

2. It incorporates the axioms in our system of work, and gives them a standard place therein.

3. It provides for the regular and legitimate use of proof by superposition, and for negative demonstrations. The prevalent dislike for these methods of proof, and the feeling of doubt on the part of many as to their full validity, are probably due to our reluctance to use such a large amount of auxiliary toolage, so large indeed that we feel that we may not have it under complete control. But our scheme instead of this occasional use under protest gives a definite though subordinate position and function to these methods of proof.

4. It also opens the way to the free use of concepts of number, and of the algebraic unknown quantity, as we shall see later.

We will now consider the more definite and immediately practical ways in which the above general considerations may be applied in teaching a pupil to solve originals. Even if this scheme be not given directly to the pupil, it is useful to the teacher in the first place in shaping or directing the work of the pupil. It opens the way to a classification or grouping of originals in a certain progressive order. This order depends in the main on the progressively more difficult kind of auxiliary quantity or geometric tools used.

1. Our scheme requires that we first ask the pupil to solve a group of originals requiring the use only of A, the simplest and most natural auxiliary quantity, and that we avoid altogether the use of B, C and D for the time being. Experience has shown that a group of exercises in proving triangles equal best supplies what is needed.

2. We next naturally take up original exercises proved by the B method. A group of exercises in proving angles equal mainly by means of the triangle of which they are a part, some use of the isosceles triangle being made, and another group in proving line segments equal in the same way, not only gives drill in the use of the B method, but serves as a review of A.

3. Both A and B may then be reviewed, and mastered from another point of view by proving exercises concerning parallel lines. Two parallel lines and a transversal form indeed but a special kind of triangle.

4. The use of number in geometry I regard as coming partly under A, and partly under D. The use of number in that it implies the internal division of an object into equal units is A; in that is uses symbolism (the number signs) external to the object treated, is D. The use of the algebraic symbol, x, also something external to the object treated or represented, I regard as essentially a form of D. But it is a kind of artificial external tool not likely to be overused by the pupil. The use of numbers and of the algebraic x not only may be incorporated into our scheme at this point without violence, but their use forms a good introduction to the geometric C and D. Hence at this point we have a group of exercises in proving the numerical properties of geometric figures, and also one in the proof of geometric properties by algebraic methods.

5. We are now ready for the method D, that is for the use of aggressive new and artificial geometric tools or auxiliary objects. Now that we do take up the use of these objects, we employ them systematically, and use one or more of them in every original. We begin by using first one auxiliary object (a line) in solving an original, then two, etc.

6. The use of D in the form of indirect demonstrations, and then of the locus naturally follows.

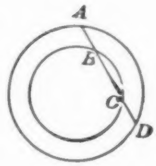
Such is the grouping of originals for Book I which is suggested by our plan. We proceed by similar graded steps in the other books of geometry.

So much for the application of aur A, B, C, D scheme in classifying originals. This much has been tested and found useful in the classroom.

The question naturally arises, can our A, B, C, D, E, X, Y method be made a help to us in other ways? There are two other ways in which I use it though rather informally. This part of the matter has not crystallized into a formal or final statement in my mind or practice.

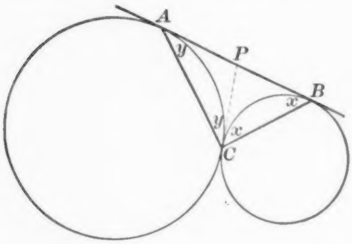
The two ways referred to are as follows:

I. The customary catalogue method of trying objects and theorems closely related to the subject matter of an original (that is, the X, Y part of the plan) may be made primary in the attention, the A, B, C, D part being merely held in the mind as a possible aid at any moment. For instance, if we have given two concentric circles with the chord AD of the outer circle cutting the inner circumference at B and C, and it is required to prove $AB = CD$, we may proceed by recalling the theorems concerning the equality of parts of a chord, and of course soon recall the theorem that the diameter of a circle perpendicular to a chord in that circle bisects the chord. If now we have our A, B, C, D scheme in mind we are in no hurry to use auxiliary quantity here, but the way being opened by the theorems recalled we have no hesitancy in drawing a diameter perpendicular to the given chord and using it as an auxiliary instrument; similarly holding our toolage schematism constantly in mind, it is natural or even inevitable for us to use at this point the axiom that if equals be subtracted from equals the remainders are equal.



II. The A, B, C, D scheme may be given the leading place and the catalogue (or X, Y) method used as something auxiliary to it. Let me illustrate this by a treatment which has occurred to me of a familiar original. We have given two circles tangent externally at C, and AB a common external tangent of the two circles, and it required to prove that the angle ACB is a right angle. If we follow the catalogue method of investigating a proof we may recall the different theorems which give methods of proving an angle a right angle and may recall that an angle inscribed in a semicircle is a right angle. If we also recall that if the common internal tangent be drawn meeting AB at P, $PA = PC = PB$, it is easy to dray a semicircumfer-

ence with P as a center and PA as a radius and prove that



ACB is a right angle. This is a rather difficult method of proof to the average pupil, and carrying as I did my A, B, C, D scheme in mind, it occurred to me that the above method of proof is essentially the D method (the semicircumference drawn through A, C, B being external to the essential part of the figure, the triangle ACB) and it also occurred to me, since the C method seems more natural and less artificial to the pupil, to inquire whether the difficulty of the situation could not be relieved by getting a C method of proof. This comes very simply from the fact that in the $\triangle ACB$

$$2x + 2y = 180^\circ,$$

$$\therefore \quad x + y = 90^\circ.$$

This plan of keeping our A, B, C, D scheme in mind and thus devising methods of proof of each of these leading kinds for the same original not only has the advantage of relieving the difficulties of particular pupils but also of throwing a double or manifold light on a principle or relation. I also find that the realization that there probably exists an internal proof makes the pupil more ready to accept and search for the more elegant external proof.

These methods just described of holding A, B, C, D and X, Y in the mind and giving now one, now the other, the leadership are especially useful in dealing with groups of miscellaneous exercises.

Allow me now to state in some detail the actual methods which I follow in this connection in the classroom, and then conclude with some statement of the specific advantages of the scheme as a whole and the classroom results which I, at least, get.

1. I think it is a considerable aid in teaching a class in geometry to work original exercises to have the class take a preliminary course of say one hour a week for six months in geo-

metric drawing, but I do not take space to discuss this matter in detail.

2. I have the textbooks theorems arranged so that the treatment of parallel lines is postponed till the use of the triangle both in text theorems and in originals has been thoroughly mastered. There are several reasons for this arrangement, but the chief one that bears on the subject in hand is that by this plan the originals solved during the first few weeks call for only the simplest methods of proof, viz.: the A and B methods.

3. After about eight or ten recitations on text theorems, the matter of original demonstrations is introduced in the last fifteen minutes of the daily recitation in an oral way.

4. One of the best ways to make my point of view clear will be, I think, to state the particular original which I always begin with, to state the difficulties which I meet with especially with the lower half of the class, to try to point out the source of these difficulties, and to state how I meet these difficulties.

The original I present first is the one referred to on p. 124. The data being as there stated, the object is to prove the triangles ABO and BOD equal or congruent. I of course ask the class to state various conditions that make two triangles equal, and a list of these is written on the blackboard. I ask the class to state whether they can point out any equal angles or any equal sides in the two given triangles. Usually the first statement volunteered is that $AO = OD$, or $AB = BD$. The reason for the fact that the class begins by searching for equal sides is that angles are implicit, and not simple, definite, palpable entities like sects; also separate lines like AO and OD make a more palpable impression than does a side like BO used twice.

If I point out to the class that no reason can be assigned justifying the statement that $AO = OD$, I am met by the reply, "But they are equal, aren't they?" That is, the pupil tries to shift the burden of proof upon me. He assumes the right to take any pair of objects as equal, provided I cannot prove that they are unequal. I break up this habit by writing each part of the hypothesis out clearly, on the blackboard, and also each step in the proof as soon as the step is made, and by asking after each statement volunteered by a pupil the two questions "Is it in the data of the proposition?" "Is it in the preceding part of the proof?" Cut off from fallacies in this way, the pupils soon

discover that $BO = BO$ by identity, $\angle ABO = \angle OBD$ because the parts of a bisected whole are equal, and $\angle AOB = \angle BOD$ because all right angles are equal and hence that the triangles are equal by the application of the familiar theorem. Simple as are these steps, the pleasure, not to say excited joy, with which class after class makes these discoveries, impresses upon me how much we continually overestimate the abilities of beginners as they enter upon this work, and how futile it is to begin this discipline with anything like elaborate systems of analysis. Of course there are individual pupils in each class who are capable of beginning with more difficult work than is suggested above, but they seem amply repaid in watching the slower progress of the lower half of the class (1) by the varied light which they get on the subject and (2) by the stimulus which comes from a realization of their superior ability. As for the instructor, the teacher's life affords no keener pleasure than the sight of minds, hitherto dull and inert, thus germinating with the beginnings of original power.

5. At this point the study of text theorems is dropped for the time being, and five or six lessons are spent entirely in proving original exercises, viz.: exercises in proving triangles equal, and in proving lines and angles equal by means of triangles.

6. After resuming the study of text theorems, more or less time is spent on originals each day in connection with the text theorems.

7. At the end of Book I two or three weeks are spent on originals alone.

8. The original exercises studied in connection with each book are treated in the same general way.

Instruction in the use of elaborate systems of analysis is taken up gradually and largely in connection with or in explanation of particular originals.

I should state at this point that in teaching the construction problem originals as the end of Book II and in connection with subsequent books, I follow a method which affords an important economy of time, and seems to have other advantages. After the pupil has written out a full and clear statement of the solution of a few problems, I have pupils make only the drawings required (leaving all small arcs and construction lines on the drawing so as to make clear that they understand what they

are doing) and to omit all verbal statements. I tell the pupil, however, to hold himself ready to make a complete statement of the solution at any time, and occasionally require him to write out such a statement. The advantages of this method of teaching original construction problems are that (1) the pupil is able to cover much more ground, (2) he prefers using drawing instruments to writing out verbal statements, and since he takes more pleasure in his work, will work more zealously and efficiently, (3) having his mind concentrated on one line of work, he succeeds in solving much more difficult problems that he is otherwise able to solve. In the last fifteen or twenty minutes of a recitation I frequently have the main part of a class make this kind of construction solution of four problems such as the following: given the sum of the legs of a right triangle and the hypotenuse, to construct a right triangle.

Returning to our A, B, C, D, E, X, Y method as a whole, with respect to the classroom results obtained by its use, I may say that we require a pupil who is to be certified as having passed the subject of plane geometry, to pass a test on the originals of each book, and also one on the originals of plane geometry as a whole. In these pass examinations the exercises given are limited to those of medium difficulty such as proving that the diagonals of a rectangle are equal, that the angles at the base of an isosceles triangle are equal, or that the apothem of a regular inscribed triangle equals half the radius. But we try so to teach the originals required of all, that the training and culture obtained by our method shall equal that formerly obtained by the few pupils who succeeded in mastering originals by other methods.

I will mention some of these cultural results in the statement which I will now make of what I think are the advantages of my plan as a whole. This statement will conclude what I have to say.

1. The classification of original exercises in groups, each group to be solved according to a general method or principle, encourages the pupil on the one hand by limiting the amount of work immediately before him. On the other hand, the sense that he is solving in any given case not merely one original but is mastering a method arouses his interest and stimulates him by giving him a sense of achievement.

2. The order of groups of originals given above, on the one hand has the positive advantage of proceeding from the simple to the complex. On the other hand, it has the equally important negative advantage of preventing the pupil from forming the habit when in difficulty of drawing a maze of auxiliary lines which only entangle and bewilder him. Most of all, it does this for the weak pupil, who is particularly likely to fall into this error, and who especially needs this safeguard.

3. The A, B, C, D, E, X, Y scheme enables us to simplify an analysis often by cutting out or eliminating a whole group of toolage objects at once.

Thus, methods A and C apply only to relatively complex data. For instance, there is no use in trying the A method or theorems requiring it when we are trying to prove the angles at the base of an isosceles triangle equal; for these angles have no internal divisions. The same remark applies to the proof required in connection with the concentric circles mentioned on p. 129.

4. It aids the memory in recalling proofs.

5. It gives some sort of a clue or start (it may be only a rude one) for every original. It ends the cry, "I don't know how to begin"; "I can't even make a start." The pupil can always begin by noting the geometric objects given (X); and the theorems relating to these or combinations of them (Y); and then proceeding according to the A, B, C, D scale.

6. It includes all kinds of proof in the pupil's logical repertory on an equal footing in proportion to their efficiency.

I had hoped to take up the discussion of methods of proof used in text theorems of geometry, and show that under our scheme each has a certain naturalness or even inevitableness, but must omit this owing to lack of time.

7. It helps coördinate the study of geometry with other studies and that in a vital way. (1) For instance, algebra is introduced, not merely because it is a help in attaining results but also because of an important similarity of toolage nature. (Thus the use of algebra symbols is a kind of D as pointed out above.) (2) The A, B, C, D classification of auxiliary quantity applies to auxiliary quantity as used in any other department of study as physics, chemistry, psychology, language or ethics. When it is properly grasped, skill acquired in its use

in any one department should be available in every other department.

8. Hence it should be a step toward giving geometry a wider, deeper, and more definite cultural value.

LAWRENCEVILLE SCHOOL,
LAWRENCEVILLE, N. J.

THE SYLLABUS METHOD OF TEACHING PLANE GEOMETRY.*

BY EUGENE R. SMITH.

In estimating the values of different methods of teaching a subject, it is first necessary to consider the purposes for which the subject under discussion is taught. Geometry has a double purpose: its *facts* are of use in many vocations, and indeed, in occasional instances, in the everyday home life of man or woman; while, on the other hand, the *methods* and *habits* of correct geometrical work—the logical reasoning from known premises to their infallible conclusions—are of inestimable value to every intelligent person, even aside from their direct application to certain of the professions, such as branches of engineering. You will notice that I have made a distinction between the use of the *facts* of geometry in such a vocation as carpentry, metal working, and other trades, and the use of its methods in the scientific professions.

If there were only the facts of geometry to consider, a man might need to know that the sum of the angles of a triangle was a straight angle, but the reasons why would be of no concern to him. He might, then, carry a convenient table of geometrical facts in his pocket, and refer to it when necessary, or even memorize the comparatively short list of propositions which are of so-called "practical" nature. Why should he waste his time studying theoretical geometry? It would be nonsense.

It is only in case geometry has an important purpose, aside from the direct applicability of its facts to vocations, that the study of the proofs of geometrical propositions should have a place in our educational system. Has it such a use? I believe that it has, and that its great value is in developing the power

* Read at a meeting of New York Section.

to think along correct lines of logical thought; if properly handled, it makes the pupils think more intensely and interestedly than any other subject fitted to pupils of the same age. In saying this I do not discredit the claims of other subjects to supply mental training along logical lines. Many other subjects *contain* logic, but geometry *is logic*. It is probably the only subject suited to immature minds in which one can be expected to see the compelling reasons back of each step taken.

Granting then, that geometry has a double purpose, do we need two methods of teaching it? That is, as yet, an unsolved question; for while the second purpose of geometry applies to everyone—and it would be a pity to deprive anyone of its valuable training—yet there are many whom lack of time, on account of the requirements of earning a living, forces to forego the broader training of the subject, and to certain of these the narrower course might be of value. These cases, however, are beyond the purpose of this paper, for the pupils under discussion are, I take it, the average secondary school students.

What method will give the pupils who take the secondary school course the most efficient training with the least waste of energy? I believe most emphatically that the syllabus method, in its best form, is the answer. It seems to me that work in which the pupil follows someone's thought, without initiative of his own, should be reduced to a minimum. But, you may ask, can you expect the pupil to work out the theorems of geometry—the results of centuries of discoveries—by himself? It is this question which has done more than any other thing to retard the development of the syllabus method, for it shows an absolute ignorance of its aims and methods. I hope to answer it by the rest of this paper.

Just a word in characterization of the three methods of teaching geometry in common use:

1. The "text-book method" is the method where the author does the thinking, and the teacher and pupils follow his lead more or less blindly, according to their several abilities. The more modern texts attempt to make up for this by urging a great deal of practice in "solving originals," the originals being any geometrical truths which the author in question did not consider of sufficient importance to be placed in the text itself. What a travesty of pedagogical psychology it is! For the im-

portant truths, the ones the pupil needs to make wholly his own, he can read the author's proofs, while the unimportant ones are given him for his own training. To my mind, this solving of originals can never give to a child the sense of logic which the development of the whole subject as an investigation does; while nothing can make up to the pupil for the time spent in, as the students too often express it, "learning the proofs." It has been estimated on good authority that perhaps 75 per cent. of the pupils studying geometry from a text get the proofs largely by memory.

It is sometimes said for the text-book method that it enables the pupil to read a great number of proofs and so familiarizes him with the subject. There is no doubt that reiteration drills the memory, but I doubt very much whether reading geometrical proofs trains the logical faculties so that a pupil can apply his training to other geometrical questions.

I do not deny that a good teacher can overcome the handicap of a text-book, but I do assert that it fetters the originality of both teacher and pupil, and that only the best of teachers should dare to use one—and then with constant care lest he become a mere copyist and drillmaster.

2. The suggestive method is the method where the author leaves part of the thinking for the teacher and pupils to develop; it seems to me to be a step in advance, in spite of some objections to it which I will discuss later.

3. The syllabus method is that in which practically all the thinking is left to the teacher and pupils; the author doing little but arranging the propositions in some good order, and indicating the places which need special attention. Notice that I have included the teacher in each of the characterizations, for I consider the teacher the key to the situation.

Before going farther, I shall explain in detail just what I mean by the syllabus method. In this method, each pupil is supplied with a book, or syllabus, which contains:

1. The definitions, axioms, etc., and a fairly comprehensive discussion of them. This part is never assigned to the class for study before it has been thoroughly taken up in class work; it is used to guide the teacher as regards order and subject-matter and as a reference book for the pupils.

2. The statements of the theorems of plane geometry, arranged in some logical order.

3. A good discussion of methods; including quite frequent references to the uses of theorems and their classification, also a very few sample proofs, illustrating the different methods of attack, and developed from the standpoint of discovery.

4. A good body of exercises, preferably grouped partly under the theorems—to save the time of the teacher—and partly in general lists, to give the pupil practise in picking out the applications of the various theorems.

5. Classified summaries of the theorems.

I have already said that the teacher should develop the preliminary definition work and its most direct consequences with the class *before* assigning it to the pupils or study. Even as early as this the teacher should begin to impress upon the pupils the two great features of geometry, and therefore of a successful method of teaching it; namely, its naturalness, by which I mean the everyday common-sense way in which each step is arrived at from known data—its inevitableness, if I may use that expression; and secondly, the possibility of grouping the facts on a basis of usefulness. This should include correlations, and the study of likenesses and distinctions, and should, when possible, group theorems of the same type under one general statement.

When a pupil grasps the following statement of method, the first great step has been gained.

Any theorem must be proved by some known truth which has the same kind of a conclusion (unless it is proved by pure logic from its converse or contrapositе, and this is usually very easy to determine), and so the logical way to start any proof is:

1. To list previous methods of obtaining this kind of a conclusion, that is, those of the same class;

2. To choose a method, using the "given" to eliminate those impossible of application;

3. To attempt to apply this method.

Suppose now that the preliminary definitions, axioms, and their immediate consequences have been thoroughly discussed, and that the facts so far obtained have been classified, so that the pupil knows, for instance, how to attempt to prove one line perpendicular to another, in what known cases angles are equal, etc., and the teacher starts the theorems. Does he, as so many seem to think, say to the pupil, " Now, here is a theorem, you go

over in the corner, turn your face to the wall, and stay there until you have proved it?" Far from it! At the beginning most propositions, and throughout the course the most difficult ones, are fully discussed in class, and this discussion is, perhaps, the most distinctive thing in the syllabus method. The teacher is naturally the leading spirit of the discussion, but he is no longer an instructor looking down upon the pupils from the heights of superior knowledge, he is a fellow investigator, and if the teacher and the class can get into the spirit of harmonious partnership in discovery, the result is bound to be successful.

The teacher starts such a recitation with a mind absolutely open; he is not bound by the fact that a certain text gives a certain proof; he is there to find, with his pupils, the easiest and best way for them to prove that proposition. Right here I wish to interpolate the fact that I have often had two classes, doing absolutely the same work, on the same day, with no apparent influence except their individualities, find different proofs for a theorem—and each declared emphatically that the other proof was not nearly so satisfactory. The proof found by each class was the natural one for them, and no other method would have fitted the proof to the class as did the syllabus method.

But by what procedure does the teacher guide the investigation? Almost entirely by questions; and in geometry, as in law, questions which imply their own answers are barred. The teacher's first questions, if a full discussion is to be gone into, ask to what class the theorem belongs, what methods of proving this kind of theorem are known, how to draw the figure, what the hypothesis is, and which of the methods listed are manifestly impossible in the given figure. Having determined the methods which may be applicable, by this method of elimination—and there is often but one remaining—the class chooses that one which, after examination, seems to be most likely to give the desired result, and the development proceeds. After some practise in this kind of work the pupils can ask these first questions themselves. The teacher always questions concerning the "why" of the next step; he never tells the step, and says, "why" afterwards, and it is here that I believe the suggestive texts fail to accomplish the best results. They also attempt the question and answer method, but, by the very nature of such

questioning, it cannot be printed and put in the hands of the pupil. Granted that the first question is correctly put, and does not imply its own answer, is not the second question almost always *based on the answer to the first question?* Evidently it is, and therefore the pupil could answer the first question from the second, and so on. However, most at least of the suggestive books, do not even attempt to ask questions which do not imply the answer, but get the form of the discovery method, without its intent, by practically stating the fact, and asking "why" afterwards, or else by giving the proof in a more or less disguised form. Let me illustrate:

Theorem—"The angles opposite the equal sides of a triangle are equal."

Book A, a suggestive text, gives the figure, with the auxiliary line, the hypothesis and conclusion in full, and goes on as follows:

"Suggestion 1. Let AM be drawn to represent a bisector of $\angle A$, and be extended until it meets BC, at M."

It does not even say "Why" after this. The construction is, as a matter of course, told to the pupil; of course, he could not find out what line to draw! Did you ever have a pupil say to you, after being shown some construction line in an original, "Of course I see the proof now, but how would I ever think of drawing that line?" The introduction of auxiliary lines is, as we all know, the key to a good share of the proofs, and only in the most exceptional cases, if ever, should the pupil be *told* what line to draw. The intersection of the bisector with the base is, of course, ignored. This would naturally come up for discussion in the syllabus method, as would the possibility of adding more conditions to the drawing of the line.

"Suggestion 2. Compare $\triangle ABM$ with $\triangle ACM$," and the authority is referred to. Suggestive indeed! "compare" the triangles, when the only thing done with triangles so far is to prove them congruent, and the reference even tells which theorem will do that.

"Suggestion 3. Compare $\angle B$ with $\angle C$," and the reference again!

"Therefore—"

Now I will admit at once that the pupil has thought a little for himself, but has he learned anything about how to prove

a theorem? Does he know why the angle was bisected, and why the triangles were proved congruent? I believe not, except in so far as he sees it for himself, for certainly the suggestions did what the old style text has always done—taught him the proof *not how to discover the proof.*

Book B, another suggestive text, does not give the figure, but the hypothesis and conclusion are given in full, in terms of the letters of the supposed figure. Rather an evasion, though it forces the pupil to draw his own figure, and in so far, is an improvement.

"Suggestion 1. Draw a line bisecting $\angle A$, and meeting BC at M."

"Suggestion 2. Compare the $\triangle ABM$ and ACM." No authority referred to.

"Suggestion 3. Compare $\angle B$ and C, using scholiums under Th. VI."

"Conclusion."

Is it very much better? Would you not say that the function of the pupil was to look up omitted references rather than to do any of the thinking?

Now consider the syllabus method of attacking the same theorem:—the pupils would decide that angles were to be proved equal, and would list the known ways of proving angles equal. They would discover that no way could apply to the figure except showing that the angles were corresponding parts of congruent triangles. To form two triangles, some line must be drawn, and it must be drawn so that $\angle B$ and $\angle C$ are in different triangles, as are the given equal lines AB and CA. Naturally the line is drawn from A. But there are several ways of determining a line; which should be used? That one which will most readily prove the triangles congruent. As any line drawn from A will make two equal sides, the included angle is the obvious necessity, and so the angle is bisected.

This is only a rough and condensed report of the discussion, which is participated in by the whole class, but it serves to show that the reason for the step is studied before the step is taken. By this method the pupil knows why he did each thing, in other words, he is learning the method of proof rather than the proof itself; *he* is reasoning—not the teacher nor the author.

Let me ask this "How many of you who use a text, sug-

gestive or otherwise, would consider it a fair question to ask your pupils why the auxiliary lines were used in any of the theorems? Would the pupils consider such a question as unheard of, or would they take it as a matter of course?" I know that some of you ask such questions, and in fact do all things which the syllabus method does, but if so, why the text, which can only cripple your efforts?

I could multiply examples like this theorem almost indefinitely, but it is only necessary for you to turn to any text, and read it in the light of what I have just mentioned, to see that I have not chosen an isolated example, but rather a simple and typical one.

But the syllabus method does not develop all work by class discussion, it asks the pupils to work out very many of the propositions with little or no assistance, even from questioning. This is sometimes done in class, where each pupil works along on the succession of propositions at a pace governed by his ability. The teacher is there to ask the illuminating question when the difficulty becomes too great—to often refuse to ask or answer when the pupil should conquer entirely by his own efforts, and in fact, by wise guidance and enthusiastic leadership to carry the efforts of all the members of the class to some accomplishment.

Again, certain theorems are assigned for work at home, and if the theorems are well chosen, and the pupils have been well trained in the methods of attack, the larger part of the class will be able to find the proofs.

It is of course true that some of the texts omit certain proofs altogether, and in so far, they are following the syllabus method. They have not, however, given the student the proper training in attack, and it can be no wonder if the pupils do not do any too well with such theorems, unless the theorems are quite simple ones.

But to go on with the method. After having a proof, a pupil may, or may not, be required to write it out in a predetermined logical form. If he does write it out, he hands it in to the teacher, who returns it with indicated corrections. The pupil then makes the corrections, and files the proofs in a notebook, which eventually becomes a complete geometry built up by the pupil. It is a well-known fact of psychology that a thing be-

comes one's own largely in the degree in which it results from one's own effort. I think this needs no comment.

The matter of requiring a pupil to keep a notebook of theorems should be determined by the conditions under which the teacher is working. A teacher who has a crowded schedule and large classes would probably sacrifice more in strength and inspiration in examining such books than would be gained by the pupils in writing the proofs. On the other hand, if the conditions are such that the teacher can do the work, the writing out of the theorems in definite concise form is a valuable training. I might say that I have always required my pupils to keep notebooks, but my classes have never exceeded twenty-eight, nor has my schedule been an overburdening one. The system is in just as successful use in some schools which require no notebooks, as in those which insist on that feature.

I wish to emphasize again that the pupils remember the theorems by classes, and by their relations to each other. To bring in psychology once more, a new concept is hard for the mind to grasp, but an addition to an established group of concepts is much more readily made part of one's knowledge.

Another feature of the syllabus method is almost daily cross-questioning, or oral review. A few typical questions follow:

What ways of proving lines parallel do you know? If you wished to prove two line sects equal, what way would you think of first? Why? Give a brief outline of a proof of the theorem for the sum of the interior angles of a polygon. A second proof. Why are triangles used? Which method of dividing the figure into triangles do you consider better and why? What is the theorem good for? (After having considered the exterior case also.) If the sum of the interior angles is five times the sum of the exterior angles, how many sides has the polygon? How would you state the converse of the interior angle theorem? the contraposite? Does either give a new theorem? etc.

You may say that such questions can be, and are, asked in any method. Yes, but such discussion is the natural outcome of the syllabus method, while it is more or less of a graft on any other. The moment the text-book is discarded, oral and individual work must take the place of the book, and so these helpful and interesting reviews and oral tests are a natural outgrowth of the method. An interesting variation is the "spelling match" with geometry questions instead of words to spell.

I have given an idea of what the syllabus method is; now I will answer a few questions and criticisms which are frequently made.

"Doesn't it take more time to cover the subject?" No. The first part takes quite a good deal longer, for the pupils must become used to the methods before very rapid progress can be made, the last part takes very much less time. I think considering amount of work covered in the whole course, more can be done by this method.

"Do you have time to do originals?" Hundreds of them—not to speak of the fact that the pupils are doing work which is largely original all the time. The originals also are done in groups when an important topic is to be studied; for example, the teacher may say, "To-day we will study the trapezoid; let us base its discussion on that of the triangle. Can you suggest any triangle theorems which could be adapted to trapezoids?" The pupils suggest certain ones, and they are investigated. In the course of the discussion, the auxiliary lines which are of most value in trapezoids are discovered, and at the end of the lesson, the pupil has a better idea of how to attack a question relating to trapezoids than he could obtain by a very large number of isolated exercises.

"Can it be used with large classes?" The questioning method of developing theorems is especially adapted to large classes, for it gives many opportunities for recitation—enough so that all can take part—and it insures that each pupil shall, to some degree at least, think for himself. As I have already said, I have never had a geometry class of over twenty-eight, but others have used the method successfully with large classes, and under none too good conditions.

"Is it not harder for the teacher?" Yes, I think it is. If you are looking for an easy time, avoid it; if for results, try it. The first thing a teacher trying the method discovers is that he doesn't know as much geometry as he supposed, for new avenues of research open all round him from day to day. That means additional work and thought for a time, but all in the line of professional advancement.

If notebooks are used, the examining of the written theorems takes time. True, that time decreases as the teacher becomes accustomed to the method, but it is never negligible.

As I have already said, there are many places where the teacher should not require this notebook work, and in such cases, a slight variation in the method would be used; for example, the pupils might be allowed to make condensed notes on the theorems as developed in class, and write out only the ones assigned for home preparation.

"Can a teacher who has had no special training use the method?" If a teacher can see geometry from the standpoint of a pupil, and can conduct the recitation as a fellow worker, he is on the highroad to success. On the other hand, one cannot have too thorough a knowledge of geometry and its allied branches, for it all helps to a more intelligent guidance of the class. Perhaps as important a requisite as any is the ability to determine the correctness of a proof.

"Do pupils, taught by this method, get into college?" With many of you this would be the supreme test. They do, and they not only get in, but they stay in—which after all is of some importance—and they do good college work because they have been taught to think.

"Do the pupils really discover the proofs for themselves?" There is hardly a theorem in plane geometry—I am not sure that there is one—which some of my pupils have not worked out for themselves. Many times a theorem which I have decided to take up in class is worked out beforehand by some ambitious students. My answer to the question of expecting the pupils to rediscover all the geometry of the ages is: 'No, I do not expect it, neither do I ask it, but nevertheless I get it in varying degrees; and, at any rate, *the pupils are thinking, they are reasoning, and their logical faculties are developing.*"

"Does it not encourage the pupils to cheat, either by using a text, or by copying notebooks?" I think not. Some pupils will be dishonest by any method, but if the method is properly worked out, the average pupil will not be dishonest. Aside from any moral standpoint, the pride in accomplishment and the pleasure that the average person takes in a puzzle, will, if properly stimulated, reduce dishonesty to a minimum. I have never found this worth considering, and I think other teachers who use the method will agree with me.

Let me give one more example of development, and with a word of conclusion, I will be through.

Take the division of a line sect internally in mean and extreme ratio. Do your pupils ever squirm when it comes along? But it is a perfectly natural proof, and if discovered in the following manner is not likely to make trouble. Let AB be the given sect, P the required point so that $AB/AP = AP/PB$. Of the four terms of this proportion, how many are known? (one, AB). If we could change the form of the proportion so that there were more known terms, would you consider it an advantage? (The pupils grasps the idea that such a change would add to his knowledge concerning the required proportion, and answer "Yes.") Now look at the figure and see if there is any relation between the terms which could be used to get a known term from unknown terms. ($AP + PB = AB$.) Is there any way to get this form in the proportion? (By composition). The proportion is then written $AB + AP/AB = AB/AP$, and it is noticed that the known term AB is used three times, while there is but one unknown term AP which is used twice. From which term must we work? (AB.) What is seen about AB? (It must be a mean.) What theorems do you know in which a sect is the mean between two others? (The students state them.) Where then ought the sects AB, and AP and $AB + AP$ to lie in the figure? (Students suggest, perhaps, the altitude to the hypotenuse of a right triangle and the sects of the hypotenuse.) A figure is drawn freehand so that the required parts are seen in the desired position. In constructing this figure, where must we start? (With the known line AB, by drawing it perpendicular to an indefinite line, say at K.) What fact about a right triangle is likely to be of assistance in constructing this right triangle? (It can be inscribed in a semi-circle.) Is anything besides the altitude known? (Part of the hypotenuse.) From what point must it be cut off? (From K.) Why? (Only known point.) The point is called L, and it is easily seen that the unknown AP must be added to each end, from K and from L, to complete the hypotenuse, and that therefore the midpoint of KL is the center of the required circle, while the radius must be the sect from this midpoint to the other end of the altitude. The circle is then constructed, and the required sect AP is found. A similar, and no harder demonstration would be found by using the tangent; I give this one because pupils so often suggest it. I leave it to

you as teachers whether such an analysis does not present the subject to the pupil in a way likely to interest and benefit him.

In conclusion, I wish to say that I am an enthusiast, but not, I hope, a crank. I try to make my pupils think rather than discover, for thinking is of more importance than the mere discovery of theorems, which should be a means, not an end. I do not claim that this method is a panacea for all ills. The slow pupil is still slow, the careless one is still careless, and sometimes the uninterested are still hard to arouse, but in each of these classes I believe that a little more of the divine fire shows. If there is any way to make geometry a student's holiday, I have yet to find it, but by this method, it is at least a live, business-like, and commonsense subject, and the pupils know that good judgment and sound logic can overcome all its difficulties.

POLYTECHNIC PREPARATORY SCHOOL,
BROOKLYN, N. Y.

INTERESTING WORK OF YOUNG GEOMETERS.

BY J. T. RORER.

I.

Although that part of geometry commonly studied in our schools has been so thoroughly worked over year after year for so many centuries, nevertheless new and interesting theorems are occasionally discovered. The following theorems which are not found in the text-books commonly used and which probably have not appeared in the journals, were discovered by a second-year high school student.

1. If the segments of the altitudes of any triangle between the vertices of the angles and the orthocenter be perpendicularly bisected, the lines when produced will form a new triangle congruent to the original triangle; also,

2. The orthocenter of the first triangle is the circumcenter of the second, and conversely; also,

3. If the adjacent vertices of the two triangles are connected, an equilateral hexagon having its opposite sides parallel, will be formed.

The above theorems were suggested to the mind of the stu-

dent who announced them, by a study of concrete geometry in connection with the demonstrative work of the text. The class had performed such routine work as drawing the altitudes, medians and bisectors of a triangle; inscribing and circumscribing circles, etc. The above theorems were the result of independent constructions made by the student and correctly interpreted by him.

The proofs are quite easy and were soon supplied by other members of the class. Many related exercises and theorems may be devised pertaining to the figure involved.

II.

The problem of trisecting an angle always proves inviting to beginners. The history of this famous problem when brought to their attention sometimes tends more to heighten than to lessen their enthusiasm.

One such young enthusiast invented the following method of trisecting. The construction approximates so closely to the desired trisection that an error can hardly be detecetd if the drawing is made on a scale commonly used in class-room work.

To trisect $\measuredangle\theta$, or its arc AB. On chord AB as diameter construct a circle, center M. Let E be the intersection of the bisector of θ and the circle—the intersection toward the vertex. Let C and D be points of trisection of the outer semicircle, *i. e.*, $AC=AM$; draw EC and ED; these will approximately trisect arc AB; let these points of approximate trisection be F and G.

The trigonometry class, with some suggestion as to the method of attack, evolved this method of finding the error of the above construction.

Let O be the vertex of $\angle\theta$; let H be the mid point of the outer semicircle; draw OF, one of the approximate trisectors of $\measuredangle\theta$. In $\triangle OEF$, let $\measuredangle OFE=\kappa$, $OF=r$; then

$$\frac{\sin\kappa}{\sin OEF}=\frac{OE}{r}$$

but, since arc $HC=30°$ by construction, $\measuredangle HEC=15°$.

$$\therefore \measuredangle OEF=165° \text{ and } \sin OEF=\sin 15°.$$

Again

$$OE=OM-EM=OM-AM=r\cos\frac{\theta}{2}-r\sin\frac{\theta}{2}.$$

Substituting these values of sin OEF and OE and solving,

$$\sin\kappa = \sin 15^\circ\left[\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right] \tag{1}$$

Let $\frac{\theta}{3} = \measuredangle AOF + \epsilon$, where ϵ is the correction to be applied to the first (or third) approximate third of the given angle, or arc, to obtain the true value, then,

$$\measuredangle AOF = \frac{\theta}{3} - \varepsilon$$

and

$$\measuredangle FOH = \frac{\theta}{2} - \measuredangle AOF = \frac{\theta}{2} - \left(\frac{\theta}{3} - \varepsilon\right) = \frac{\theta}{6} + \varepsilon$$

Again,

$$\measuredangle FOH + \kappa = 15^\circ \quad \text{(Exterior angle of } \triangle\, FOE\text{)}$$

$$\therefore \frac{\theta}{6} + \varepsilon + \kappa = 15^\circ$$

$$\therefore \varepsilon = 15^\circ - \left(\frac{\theta}{6} + \kappa\right) \tag{2}$$

From equations (1) and (2), the value of ϵ may easily be computed for any value of θ. The results of such computation are tabulated for each 10° of arc.

θ	10°	20°	30°	40°	50°	60°	70°	80°	90°
	−16′.5	−27′.2	−32′.7	−33′.9	−31′.5	−26′.2	−18′.6	−9′.7	0′

θ	100°	110°	120°	130°	140°	150°	160°	170°
ε	+9′.7	+18′.6	+26′.2	+31′.5	+33′.9	+32′.7	+27′.2	+16′.5

It will be seen that these corrections are about within the limits of accuracy for ordinary class-room "compass and ruler" work—whether pencil, ink or chalk be used.

Central High School,
Philadelphia, Pa.

THE SEVENTH TO TENTH GRADES A UNIT IN MATHEMATICS.

By R. L. Short.

Since this paper is intended for educators interested in public school work, it may be proper to first define the topic. This may be done best by giving an explanation of present conditions in our elementary and secondary courses.

The present course is:

Seventh and eighth grades.........	1 unit.
Ninth grade......................	1 unit.
Tenth grade......................	1 unit.

The paper is written to call attention to the defects of the course as now given and to raise the question: How may these defects be remedied?

For the past five years mathematics associations have taken up the cry for better mathematics teaching. Professor Perry, of England, offered a remedy by suggesting the teaching of physics and chemistry in algebra and the use of many measurements and experiments in the mathematics class. As a result, the disciples of this movement went up and down the land demanding correlation. But where are those disciples? You can't find them now. There was a fundamental error in the scheme. The material given the pupil to *use* in mathematics was beyond him, often beyond his comprehension. The result was the student was so confused by the mass of new facts and new surroundings that he got no science and no mathematics.

We all teach some physics notions—it's the fashion—and a little such is well. But the teacher who tries to lead his pupil to discover and develop physics laws which cost the life endeavor of some of the greatest intellects this world has known and to discover the equations governing those laws, certainly harms the pupil, the subject and the cause of education.

There is a relation, a simple one, one that is easily within the grasp of every pupil, that has been entirely overlooked. With Professor Perry, I firmly believe that mathematics should not be set off in compartments. However, I am of the opinion that the articulation must be made with *known subjects* rather than with *unknown subjects.*

In the grammar school we teach arithmetic—just arithmetic—such as has been taught in grades IV and V. So far as my observation goes there is little or no thought work done, no development of principles, no application made. In the ninth grade we teach algebra—just algebra. No application is made; the arithmetic does not help the algebra nor does the algebra help the arithmetic; geometry is not anticipated. In grade X we teach geometry. Little or no use of algebra is made, the equation is neglected, and pure reason introduced. Arithmetic is too far in the past to even get a hearing.

Such teaching during these four years is, to my mind, sheer waste of time. A few illustrations may strengthen the assertion. Your seventh-grade pupil, your eighth-grade pupil, ninth-grade pupil, tenth-grade pupil knows nothing of the constitution of number, of the relation of numbers, of the use of number. He knows a few combinations of numbers. He knows a few rules and no *reasons why*. He may know $6 \cdot 8$ but when asked for $6 \cdot 18$ he frantically grasps a pencil or crayon. He may know the square of 32 but $16 \cdot 64$ is impossible without pencil and paper. He knows that $a^2 \cdot a^3 = a^5$ but he cannot tell you what power of 3 is $9 \cdot 27$. He will tell you that the square on the hypotenuse equals the sum of the squares on the two legs of the triangle but he cannot tell you whether the diagonal of a square whose side is 8 is 14, 15 or 16, even though he may possibly know that diagonal to be $8\sqrt{2}$. To multiply a fraction he is not quite sure whether to multiply the numerator or the denominator of the fraction by an integer, so to make certain that he does all that he should do he multiplies both. He knows the square of $(a+b)$ but he cannot tell you the square of $30\frac{1}{4}$. None of these defects are the fault of the child. They are chargeable directly to the teacher and to the courses offered. Now for the remedy. A closer knitting together of arithmetic, algebra and geometry, an articulation, if you please, of these subjects and an understanding of principles rather than of rules will settle these troubles.

For example, when your grammar school pupil knows that $c(a+b) = ac + bc$, he should also know that $6 \cdot 18 = 6(10+8)$ and that $16 \cdot 17$ is also the product of a binomial and a monomial, namely $16(10+7)$. Mental arithmetic, rapid and accurate, is then possible. When your pupil knows how to write the product

of $16 \cdot 17$, $(3x - 17)(5x + 16)$ needs no pencil and paper. You teach $32 \cdot 32 = 1024$ but do you teach $2^5 = 32$, $2^4 = 16$, $2^6 = 64$? You teach $(a^5)^2 = a^{10}$, also $a^4 \cdot a^6 = a^{10}$. Do you teach $(2^5)^2 = 2^{10} = (32)^2$ and $2^4 \cdot 2^6 = 2^{10} = (32)^2 = 1024$?

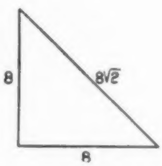
$a^2 \cdot a^3 = a^5$. Does that mean to your pupil $3^2 \cdot 3^3 = 3^5$? That is, is $9 \cdot 27 = 3^5$?

$(a^2)^3 = a^6$ and $\sqrt{a^6} = a^3$. Does $(9)^3$ means 3^6 and does

$$\sqrt{9^3} = \sqrt{3^6} = 3^3 = 27?$$

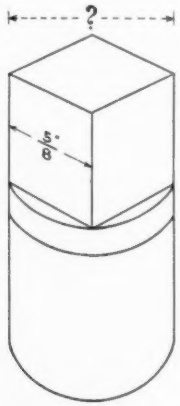
The side of a square is 8. The diagonal is $8\sqrt{2}$. Does your pupil have *any* idea of how long a line $8\sqrt{2}$ is? Does he know $8 \cdot 14$? Does he know that $8\sqrt{2} = 8(1.4+)$?

①



The stock for milling machine purposes is usually cylindrical. Try your pupil on this one: A piece of iron is to be milled so that one end is a square $\frac{5}{8}''$ on a side. What size stock must be selected? (See diagram.)

②

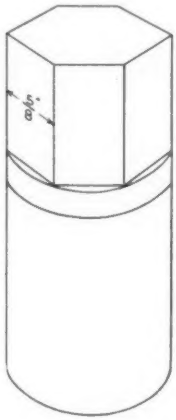


The problem is simply the application of the relation between the diagonal of a square and its side.

Now put the same question to the pupil making the end hexagonal, $\frac{5}{8}''$ on one side.

Have you taught him why multiplying the numerator of a fraction multiplies the fraction or did he just learn the rule? Does he know that a mixed number is a binomial and that there

③



are two kinds—the $a + b$ and the $a - b$—or does he square $30\frac{1}{4}$ by reducing to $121/4$ and squaring both numerator and denominator? I would suggest that the square of a mixed number be obtained mentally by means of say

$$(30\tfrac{1}{4})^2 = (30 + \tfrac{1}{4})^2 = (900 + 15 + \tfrac{1}{16}).$$

In reducing fractions to the lowest common multiple do you permit your pupils to follow this process:

$$\frac{5}{48} + \frac{11}{36} + \frac{8}{45} + \frac{7}{54} = \frac{\qquad}{2,160}$$

2)	48	36	45	54
2)	24	18	45	27
3)	12	9	45	27
3)	4	3	15	9
	4	1	5	3

Or do you combine algebra and arithmetic in this manner?

$48 = 2^4 \cdot 3$
$36 = 3^2 \cdot 2^2$
$45 = 3^2 \cdot 5$
$54 = 3^3 \cdot 2$

Select for your l. c. m. each factor which appears, giving to each the highest exponent appearing in any number concerned—*i. e.*, $2^4 \cdot 3^3 \cdot 5$.

$2^4 \cdot 3^3 \cdot 5 \div 48$ is
$2^4 \cdot 3^3 \cdot 5 \div 2^4 \cdot 3 = 3^2 \cdot 5$.

The first numerator is then $3^2 \cdot 5^2$.
Similarly the second numerator is $2^2 \cdot 3 \cdot 5 \cdot 11$.
The third numerator, $2^4 \cdot 3 \cdot 8$ or $2^7 \cdot 3$.
The fourth numerator, $2^3 \cdot 5 \cdot 7$.

Then

$$\frac{5}{48} + \frac{11}{36} + \frac{8}{45} + \frac{7}{54} = \frac{3^2 \cdot 5^2 \cdot + 2^2 \cdot 5 \cdot 11 \cdot 3 + 2^7 \cdot 3 + 2^3 \cdot 5 \cdot 7}{2^4 \cdot 3^3 \cdot 5}$$

$$= \frac{225 + 660 + 384 + 280}{2160}$$

$$= \frac{200 + 600 + 300 + 200 + 20 + 60 + 80 + 80 + 5 + 4}{2160}$$

All of which may be performed mentally.

Such a course will not only help the arithmetic but will be of infinite service in the manipulation of algebraic fractions and in the reduction of surd polynomials.

e. g., $\sqrt{(a^2 - a - 12)(a^2 + 2a - 3)(a^2 - 5a + 4)}$.

A similar articulation and application of subjects should extend through algebra and geometry. It may necessitate some rearrangement of courses in both grammar school and high school. But is it not worth while? Will it not teach the pupil to think? Will it not give him some opportunity to estimate his answer and to know within some limit at least how nearly right he is? Will such articulation not bridge over the chasm between grammar school and high school and save some of that wholesale slaughter that now occurs in the freshman year?

It is my opinion that such course can be carefully and successfully worked out. The mathematics for college, for business, for scientific lines will then be built on the foundation laid in the third to sixth grades, will be one continuous course, the various phases of secondary mathematics being brought in naturally and as rapidly as the pupil can assimilate the material. Usable mathematics can be taught and that is the only kind of mathematics that belongs to the pupil.

Technical High School,
Cleveland, Ohio.

THE EFFORT TO MAKE ALGEBRA YIELD FRUIT.

By W. A. Cornish.

There is a great searching of heart among the teachers of algebra. The idea is abroad that algebra is the barren fig tree. Our association has a committee at work revising the old or devising a new algebra syllabus. Only about a year ago the Central Association published the report of its committee on algebra in the secondary schools, which report constitutes an algebra syllabus with many suggestions as to method of treatment.

Only a few years ago the American Mathematical Society put out a new syllabus. Various other associations of teachers and scientists up and down the world have within a few years put forth new schemes both as to subject matter and method of teaching algebra. This and other similar societies at their principal meetings and at their section meetings are continually discussing new syllabuses and other things new and old having reference to the teaching of algebra. There is great unrest among the teachers of algebra. Out of what uneasiness of consciousness or conscience does all this disturbance come?

Moreover the dissatisfaction with present conditions isn't all ours. I remember that at a N. E. A. meeting not long ago a distinguished educationist whose interest does not lie especially in mathematics made the remark that, as things are, the benefits derived by high school students from the study of algebra do not begin to be commensurate with the time and toil devoted to the study, and he would dispense with the most of the algebra of the high school course and so save time for studies that would amount to something. There are many others of his opinion; among them many of our own pupils in algebra, both of this present time and times gone by.

It is, we know, a very frequent source of wonder among them what in the world it is all for, this manipulation of polynomial literate expressions. They multiply them, they divide them and factor them and extract their nth roots and raise them to the nth power and find their conjugates and perform a variety of other experiments vulgarly called stunts with them. They sometimes learn to play the game according to the rules pretty

fairly well too and sometimes get 90 and sometimes less than 90 in regent's and other examinations. But what good has it done them? What relation does it bear to life and the actual business of life? There are very, very few of them to whom their algebra becomes a permanent intellectual interest as their history, literature, economics and natural science do become. We have to admit that algebra does not, as a matter of fact, do much for many of our pupils in any of these ways. It might. It ought to. But, as a matter of fact, it doesn't.

Well then, why cumbereth it the ground? Keep algebra in the course for those who are going on to engineering courses or higher mathematics, but for the rest dispense with it. Save time for other, more important or more fruitful studies as the educationist quoted above suggested.

Shall we consent to such an arrangement? No indeed! Not one of us or any other mathematics teacher in the Middle States and Maryland, or outside that particular Eden, will consent to anything of the sort. It may be that as a matter of fact all of our pupils are not getting all the good they should in the way of fruit from this particular fig tree, but let it be this year, till we have dug around it and till our new syllabuses are in operation and there will be a crop that will astonish the natives.

Hence the activity of our committees on new syllabus, and hence the never failing interest in papers on the subject at association meetings.

Now, as to the cause of the trouble, the reason for the failure of algebra, it lies, so we are agreed, those of us who agree that there is trouble, in the fact that, so far as the experience of many of our pupils is concerned, algebra does not have enough of bearing upon the business or interests of life. And the reason for this aloofness of algebra from life appears to be its great aloofness from life, or its extremely abstract character.

Number itself is an abstraction and the abstraction consists in the detachment of the idea of quantity from the concrete whole that includes quality as well as quantity.

Then we abstract from the idea of number everything that looks towards definiteness of abstraction and dropping that and concerning ourselves only with the indefinite abstract remainder we raise it to the *n*th power and multiply the result by the *n*th

power of some other equally indefinite abstraction. Then we combine many such into polynomials and with such insubstantial material build houses of the sort not made with hands, and then we proceed to keep house in them and play games in them along with gods and archangels. And all this is great sport for us and for some of our pupils. But there is the ignobile vulgus, the common people, whom God has created in considerable numbers, who can not be quite happy unless their legs are long enough to touch the earth, and these do not find the game either interesting or edifying.

All of our committees and syllabus makers are thus diagnosing the case and are trying to remedy the trouble by making algebra more concrete.

The new Prussian Syllabus lays stress on concrete beginnings, on graphic methods, on deferring the more abstract phases of the subject, and on applications throughout to the affairs of practical life.

A commission representing eight principal learned and scientific associations of Germany which has made a report since the adoption of the new syllabus proposes a new curriculum which defers to a still later period the more abstract topics and methods, utilizes graphic methods throughout, and especially and above all insists that algebra and all mathematics are to have beginning and end and entire intervening career in connection with life and nature. Mathematics is of importance chiefly because it gives exact cognition of nature.

The report of our brethren of the Central Association, recently fulminated, thunders loud with the same thunder.

Twelve topics including the most of the complicated polynomials of algebra are expelled from the first year's work of the high school altogether. The most of the work of this year is to be problem work of a very concrete kind having its origin in well known or easily derived principles of arithmetic, geometry and physics. Multiplication and factoring are to be associated and are to lead straightway to problems involving quadratic equations and simultaneous equations. Proportion is to be taught early. Fractions, excepting such as the child may be assumed to be familiar with in connection with arithmetic, are taught late. Algebra is to be the tool of nature study. Those things that make it a useful tool for nature study are to be emphasized.

All of these proposals of reform in method and subject-matter of elementary algebra are in the same direction and they imply the reasons why it is impossible to contemplate the abandonment of algebra as an essential element of the high school curriculum.

Mathematics has been the great tool by which man has raised himself from the condition of the ignorant, impotent, superstitious wretch that he was in primæval state to that of the noble lord of creation that we see about us. And as mathematics has been in the history of civilization, the tool that has conquered the world, so it is in education the key to the interpretation of the world.

W. T. Harris' description of the field of mathematics and of the relations of its several branches to each other and to the world is useful for those of us who are teachers of elementary mathematics and who need to keep our teaching concrete and fast-rooted in the world of nature. It may be summarized as follows: Mathematics is the science of quantity and therefore the science of measurement, since measurement is the process of finding the answer to the question quantus.

If all things could be measured by the simple process of taking a unit and laying it off on the quantity to be measured, there would be no occasion for the development of such a science as mathematics. But it is impossible to measure some things that way and very inconvenient to measure other things that way. Hence mathematics has a reason for being and hence also mathematics may be defined as the general gathering of devices for performing more or less difficult jobs in measurement. The things to be measured are the extra-mental entities times, space, matter, force. There is, of course, doubt among the philosophers as to whether these things or all of them are in fact extra-mental, but they seem to be and that's enough for us.

The primary branches of mathematics are such as geometry and mechanics which formulate the principles of form and motion in accordance with which actual measurements must be made and computations carried on. The secondary or ultimate branch of mathematics is arithmetic which makes the computations and completes the measurement.

But sometimes it is impossible to get the data that geometrical or physical principles call for and the solution of our problem

must be approached from indirect data. For example, geometry tells us that the area of a triangle is one half the product of base and altitude. But there is a lion in the way that prevents us from measuring the altitude. We may, however, measure the three sides. Then there must be a mighty transformation of geometric principles, a factoring and eliminating and more factoring and squaring and substituting and rationalizing of denominators and extracting of square roots and then the principle that $T = \frac{1}{2}bh$ becomes $T = \sqrt{s(s-a)(s-b)(s-c)}$.

So there is a place for an intermediate branch of mathematics between geometry and mechanics on the one side and arithmetic on the other, namely the science of the transformation of functions. This is algebra and, conversely, this is the thing that algebra is.

The subject-matter of algebra consists either of problems in which the data are the assumptions or other principles of geometry or physics, which functions must undergo more or less of transformation before numerical evaluation is possible, or it consists of chapters of development of the principles of transformation with abundant exercises, which chapters and exercises are of value only as they give the student power to perform similar transformations in the course of the solution of actual problems of measurement. If in the investigation of these principles we come to points where we catch glimpses of things below the minima or above the maxima of mundane things, imaginaries and fourth dimensions and that sort of thing, which deal not with measurement in this world but perhaps in some other, that fact rather verifies than overturns the definition of algebra given above. We at once consign those branches of the subject to the university members of our association, the people who are doing investigating off on the frontier of the science, with instructions to keep the roadway connecting our association with the world of gods and archangels well paved and in good repair. So will they earn their bread and butter while they remain with us.

(To be continued.)

NOTES AND NEWS.

Dr. J. T. Rorer, of the Central High School, Philadelphia, has been asked to organize and direct the mathematical courses of the new William Penn High School for Girls, of that city, which will comprise both academic and vocational departments. Dr. Rorer has accepted the election and will enter upon his new duties in September.

A Meeting of the New York Section was held at the High School of Commerce, Friday, April 16, 1909, at 8:00 p. m., at which time the following program was carried out:

1. Report of Secretary.

2. Final Report of Committee on Marking Mathematical Papers, by Dr. A. Latham Baker, Manual Training High School, Chairman.

3. General Topic for the Evening: The Teaching of Geometry. (*a*) "The Use and Abuse of Text-books, Applied Particularly to Geometry," by Dr. William J. Milne, President State Normal College, Albany, N. Y.; (*b*) "Teaching Geometry with a Syllabus Only," by Mr. Eugene R. Smith, Head of Department of Mathematics, Polytechnic Preparatory School, Brooklyn; (*c*) discussion to be led by the following: Miss Grace M. Peters, Normal College, New York City, and Mr. William R. Lasher, Erasmus Hall High School, Brooklyn; (*d*) General Discussion. The old officers were reëlected and Mr. Philip R. Dean, of Curtis High School, was elected a member of the executive committee.

In the case of geometry, there has been but one period in all its history—the period of *logical demonstration*. For two thousand years the demonstrations have been polished and refined, the abstract reasoning has been analyzed and synthesized, till geometry has come to be regarded as the one perfect subject in the high school curriculum, so purified and crystallized, indeed, that it will almost teach itself.

It is more venerable than algebra, and reverence for age would well nigh deter us from raising any question as to its legitimate standing in its present form in the school program. But our complaisance has been disturbed by some developments in the new psychology and the rise of the sciences to a position

of importance in the curriculum. We had come to believe, as an inheritance of the ages, that nothing whatever could take the place of geometry for mental discipline; even now we are not prepared to capitulate this stronghold entirely, and yet we are compelled to admit that work in some of the sciences has come to occupy a very strong position in this regard, and that the chances for the development of mental power outside of geometry are much greater to-day than they were before the dawn of the present age of science.

We had also fondly cherished the traditional belief that geometry is a tonic for all mental shortcomings (and we do not propose to retreat completely from this position), but we do not now dare to place as much confidence as heretofore in the theory that wits sharpened on geometry will therefore present a keen edge in all other phases of mental activity. We have seen so many good geometricians who seemed to possess small logical sense in other affairs and so many good logicians who have small knowledge and little facility in geometry, that we can no longer believe in the inevitable transference of power gained in the study of geometry to effective use in other lines. The fact is that geometry must hold its place in the curriculum on the same basis as all other subjects. The right of eminent domain based upon the sole dictum of *mental discipline* is no longer effective. For centuries Latin held its own chiefly on this ground, but to-day it would be on the way "down and out" if its supporters had not already given timely recognition to the claims of the new education and effected a radical reorganization in the teaching of the subject. As it is, we see this ancient language even gaining ground in the present secondary curriculum because the form of its study has been, and is still being, readjusted so as to give it lively contact with every language phase of modern life. It is not simply that Latin has been found worthy to meet certain important needs of a utilitarian age, but in readjusting itself to these needs it has exemplified the new education and brought to light a more profound pedagogy, in that the *old* claim of mental discipline is quite as fully met by the *new* methods of presentation and application.—Extract from paper by Professor H. E. Slaught.